to consider the reciprocal of the distance between two bodies when their mutual attractions are to be determined, and the best practicable method of finding these attractions is generally the expansion of this reciprocal into a periodical series, mathematicians were immediately confronted with the problem of finding the co-efficients of the sines and cosines of this series. D'Alembert (*Jean le Rond*) saw in 1743 that all the $b_s^{(i)}$ co-efficients could be found as soon as the first two were known, a relation which is expressed in equation (2). Euler and Clairaut made many improvements in the actual numerical determination of these co-efficients. The elegant method of resolving the general expression of the reciprocal, $(1 - 2a\cos\theta + a^2)^{-s}$, into two binomial factors by means of imaginary quantities, which is employed by Laplace, is due to Lagrange. It is probable that the transformation of the elliptic integral which has been used above was suggested to Lagrange by the work of Landen. It has since been employed by several writers, and among others by Professor Woodhouse.*

In looking at the great amount of labor that has been done on the important problem of developing the perturbative function, one cannot fail to notice the variety of the symbols that have been introduced. This variety has become so great as to cause confusion, and in order to be sure of his formulæ it is now almost necessary that every astronomer should have a development of his own. This condition seems to be unfortunate, since it causes a repetition of work that should be done once for all.

U. S. N. Observatory, Nov. 5, 1886.
Communicated by the Superintendent.

ON THE FORM AND POSITION OF THE SEA-LEVEL AS DEPENDENT ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED WITH RESPECT TO A RADIUS OF THE EARTH'S SURFACE.

By Mr. R. S. Woodward, Washington, D. C.

[CONTINUED FROM VOL. II, PAGE 131.]

18. We proceed now to the determination of the constants V_0 of equation (6) and U_0 of (56) and (57). It has already been stated that these constants are to be determined from the condition of equality in volumes contained by the disturbed and undisturbed surfaces, a condition whose analytical statement is

$$2\pi r_0^2 \int_0^\pi v \sin\alpha\, d\alpha = 0.$$

Philosophical Transactions, London, 1804.

Substituting the value of v from (6) in this, there results

$$\int_0^\pi V \sin a\, da - V_0 \int_0^\pi \sin a\, da = 0,$$

whence
$$V_0 = \tfrac{1}{2} \int_0^\pi V \sin a\, da. \tag{58}$$

The easiest way to evaluate the integral in this equation is to substitute the value of V from equation (54). We get then at once

$$V_0 = 4 r_0 h \rho \pi \sin^2 \tfrac{1}{2}\beta \,.\, \tfrac{1}{2} \int_0^\pi \sin a\, da$$

$$= 4 r_0 h \rho \pi \sin^2 \tfrac{1}{2}\beta, \tag{59}$$

since all terms of the series, except the first, vanish in the integration. In a similar manner it may be shown that

$$U_0 = Y_0 = 2 \sin^2 \tfrac{1}{2}\beta. \tag{60}$$

V_0 may also be derived thus: For points within the perimeter of the attracting mass replace V in (58) by V_1 of (20), and for points outside that perimeter replace V in (58) by V_2 of (21). Making these substitutions there results

$$V_0 = 2 r_0 h \rho \left(\int_0^\beta I_1 \sin a\, da + \int_\beta^\pi I_2 \sin a\, da \right).^* \tag{61}$$

Substituting the value of I_1 from (22),

$$\int_0^\beta I_1 \sin a\, da = 4 \int_0^{\frac{1}{2}\pi} \frac{d\gamma_1}{\sin^2 \gamma_1} \int_0^\beta \left(\frac{\sin^2 \frac{1}{2}\beta - \sin^2 \frac{1}{2}a \sin^2 \gamma_1}{1 - \sin^2 \frac{1}{2}a \sin^2 \gamma_1} \right)^{\frac{1}{2}} \sin^2 \gamma_1 d(\sin^2 \tfrac{1}{2}a).$$

*A more direct process than that followed in the text is indicated thus:—

$$\int_0^\beta I_1 \sin a\, da + \int_\beta^\pi I_2 \sin a\, da = -\Big[I_1 \cos a \Big]_0^\beta + \int_0^\beta \cos a \frac{dI_1}{da}\, da, - \Big[I_2 \cos a \Big]_\beta^\pi + \int_\beta^\pi \cos a \frac{dI_2}{da}\, da.$$

Put
$$t^2 = \frac{\sin^2\frac{1}{2}\beta - \sin^2\frac{1}{2}\alpha\sin^2\gamma_1}{1 - \sin^2\frac{1}{2}\alpha\sin^2\gamma_1}.$$

Then the last integral becomes

$$-4\cos^2\tfrac{1}{2}\beta\int_0^{\frac{1}{2}\pi}\frac{d\gamma_1}{\sin^2\gamma_1}\int_{t_1}^{t_2}\frac{2t^2dt}{(t^2-1)^2}$$

$$= 4\cos^2\tfrac{1}{2}\beta\int_0^{\frac{1}{2}\pi}\left(\frac{t_2}{t_2^2-1} - \frac{t_1}{t_1^2-1} + \tfrac{1}{2}\log_e\frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right)\frac{d\gamma_1}{\sin^2\gamma_1}$$

in which
$$t_1 = \sin\tfrac{1}{2}\beta,$$
$$t_2 = \frac{\sin\frac{1}{2}\beta\cos\gamma_1}{\sqrt{(1-\sin^2\frac{1}{2}\beta\sin^2\gamma_1)}}.$$

Substituting these limits in the non-logarithmic part, there results

$$4\sin^2\tfrac{1}{2}\beta\left(\int_0^{\frac{1}{2}\pi}\frac{d\gamma_1}{\sin\frac{1}{2}\beta\sin^2\gamma_1} - \frac{1}{\sin\frac{1}{2}\beta}\int_0^{\frac{1}{2}\pi}\sqrt{(1-\sin^2\tfrac{1}{2}\beta\sin^2\gamma_1)}\,.\,\frac{\cos\gamma_1}{\sin^2\gamma_1}d\gamma_1\right)$$

$$+ 4\sin^2\tfrac{1}{2}\beta\left(\tfrac{1}{2}\cot^2\tfrac{1}{2}\beta\int_0^{\frac{1}{2}\pi}\log_e\frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\cdot\frac{d\gamma_1}{\sin^2\gamma_1}\right).$$

Integrating by parts all terms of this expression except the first, we get

$$4\sin^2\tfrac{1}{2}\beta\left\{\begin{array}{l}\dfrac{(1-\sin^2\frac{1}{2}\beta\sin^2\gamma_1)^{\frac{1}{2}}}{\sin\frac{1}{2}\beta\sin\gamma_1} - \dfrac{\cot\gamma_1}{\sin\frac{1}{2}\beta}\\ + \operatorname{arc\,sin}(\sin\tfrac{1}{2}\beta\sin\gamma_1)\\ - \tfrac{1}{2}\cot^2\tfrac{1}{2}\beta\left(\log_e\dfrac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right)\cot\gamma_1\\ - \cot^2\tfrac{1}{2}\beta\operatorname{arc\,sin}(\sin\tfrac{1}{2}\beta\sin\gamma_1)\end{array}\right\}_0^{\frac{1}{2}\pi}.$$

This gives
$$\int_0^\beta I_1\sin\alpha\,d\alpha = 2(\sin\beta - \beta\cos\beta). \qquad (62)$$

The second integral in (61) becomes, by substitution of the value of I_2 from (25),

$$\int_\beta^\pi I_2 \sin a\, da$$

$$= 4 \sin^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \cos^2 \gamma_2 d\gamma_2 \int_\beta^\pi \frac{d(\sin^2 \frac{1}{2}a)}{\sqrt{(\sin^2 \frac{1}{2}a - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}}$$

$$= 8 \sin^2 \tfrac{1}{2}\beta \int_0^{\frac{1}{2}\pi} \left(\cos^2 \gamma_2 - \frac{\sin \frac{1}{2}\beta \cos \gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}} + \frac{\sin \frac{1}{2}\beta \sin^2 \gamma_2 \cos \gamma_2}{\sqrt{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)}} \right) d\gamma_2$$

$$= 4 \sin^2 \tfrac{1}{2}\beta \left\{ \begin{aligned} &\gamma_2 + \tfrac{1}{2} \sin 2\gamma_2 - 2 \operatorname{arc\,sin} (\sin \tfrac{1}{2}\beta \sin \gamma_2) \\ &- \frac{(1 - \sin^2 \frac{1}{2}\beta \sin^2 \gamma_2)^{\frac{1}{2}} \sin \gamma_2}{\sin \frac{1}{2}\beta} \\ &+ \frac{\operatorname{arc\,sin} (\sin \frac{1}{2}\beta \sin \gamma_2)}{\sin^2 \frac{1}{2}\beta} \end{aligned} \right\}_0^{\frac{1}{2}\pi};$$

whence
$$\int_\beta^\pi I_2 \sin a\, da = 2\, (\pi \sin^2 \tfrac{1}{2}\beta - \sin \beta + \beta \cos \beta). \tag{63}$$

The sum of (62) and (63) is $2\pi \sin^2 \frac{1}{2}\beta$, which, substituted in (61) gives the same value for V_0 as (59).

19. By reference now to equations (3), (6), (20) to (23), and (59), we find for the equation of the disturbed surface, when the effect of the rearranged water is neglected,

$$v = 3h \frac{\rho}{\rho_m} \left(\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta \right). \tag{64}$$

The corresponding expression in polar harmonics is, see equations (56), (57), and (60),

$$v = \tfrac{3}{2}h \frac{\rho}{\rho_m} \sum_{i=0}^{i=\infty} f_i (\cos a)\, F_i (\beta). \tag{65}$$

Under the assumption that the water covers the whole sphere and is free to adjust itself as stated in §17, the equation to the disturbed surface is

$$v + \Delta v = 3h \frac{\rho}{\rho_m} \left(\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta \right) + \tfrac{9}{2}h \frac{\rho}{\rho_m} \sum_{i=0}^{i=\infty} \frac{f_i (\cos a)\, F_i (\beta)}{(2i + 1) \frac{\rho_m}{\rho_w} - 3}. \tag{66}$$

20. The position of any point on the disturbed surface is thus defined by the co-ordinates v and I, or rather v and a, since I is a function of a; v being the elevation or depression of the point relative to the spherical surface, and a the angular distance of the point from the axis of the attracting mass. I is to be computed from (22) or (23), or their equivalents (24) and (25), according as the point is within or without the perimeter of the disturbing mass. The functions which enter the last term of (66) are given by (45) and (47) respectively.

The general nature of the disturbed surface, when the effect of the rearranged water is neglected, is evident from (64). It is symmetrical with respect to the axis of the attracting mass. It lies without or within the spherical surface of reference according as I is greater or less than $\pi \sin^2 \frac{1}{2}\beta$. The values of I for a point of the disturbed surface at the axis of the attracting mass, along its border and opposite the centre of the mass, are given by equations (26), (28), and (30) respectively. If we denote the corresponding values of v by the suffixes 1, 2, 3, we get

$$\begin{aligned} v_1 &= 3h \frac{\rho}{\rho_m} (\sin \tfrac{1}{2}\beta - \sin^2 \tfrac{1}{2}\beta), \\ & \qquad a = 0 \\ v_2 &= 3h \frac{\rho}{\rho_m} \left(\frac{\beta}{\pi} - \sin^2 \tfrac{1}{2}\beta\right), \\ & \qquad a = \beta \\ v_3 &= 3h \frac{\rho}{\rho_m} (2 \sin^2 \tfrac{1}{4}\beta - \sin^2 \tfrac{1}{2}\beta). \\ & \qquad a = \pi \end{aligned} \tag{67}$$

The meaning of these equations may be most readily understood by reference to Figure 3. Thus if the circle $EFGH$ represent (in cross-section) the undisturbed sea-level surface of the earth, and a stratum of matter, as an ice cap, $ABDGE$ be added thereto, the new sea-level surface will assume the form indicated by the dotted line. The values of v_1 and v_2 as shown in the diagram are positive, while the value of v_3 is negative. If on the other hand we suppose the space $A'B'D'GFE$ to be filled with matter of less density than the average density of the earth's crust, as is the case in a lake basin, the disturbed or new sea-level surface would fall within some portion PFQ of the undisturbed surface and outside the remaining portion PHQ; i. e. v_1 and v_2 would be negative and v_3 positive.

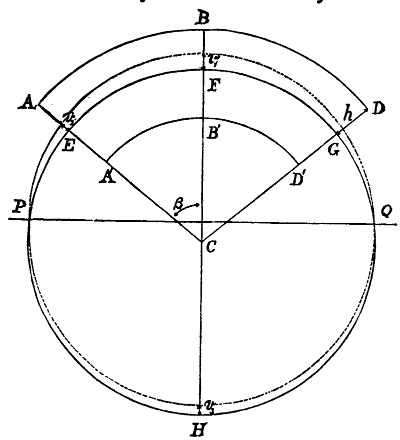

Fig. 3.

21. It is of interest to inquire what angular extent of mass will produce numerical maxima of v_1, v_2, and v_3, supposing the thickness h, and the densities ρ_m and ρ constant. By means of the usual criteria it is readily found that

$$\begin{aligned} v_1 &= \text{a maximum for} \quad \beta = 60^\circ, \\ v_2 &= \text{a maximum for} \sin\beta = 2/\pi, \quad \text{or} \quad \beta = 39^\circ 32', \\ v_3 &= \text{a maximum for} \quad \beta = 120^\circ. \end{aligned} \tag{68}$$

22. A glance at equation (66) suffices to show that the effect of the free water, if it covers the whole earth, is simply to produce an exaggeration of the type of surface defined by (64) and (65). The series in the third member of (66) expressing this exaggeration is rapidly converging on account of the diminishing factor

$$\frac{\frac{3}{2}}{(2i+1)\dfrac{\rho_m}{\rho_w} - 3},$$

which is, since ρ_m / ρ_w is about $\frac{11}{2}$,

$$\frac{3}{22i+5}.$$

The essential features of the disturbed surface are, therefore, in any case defined by (64) or its equivalent (65); and in most cases the effect of the rearranged water may be neglected as unimportant or as of no greater magnitude than the uncertainties inherent in the data for actual problems.

23. The equations (67) define the position of the disturbed surface in some of its most characteristic points. To define its position at any other point we must evaluate the elliptic integral I_1 or I_2 which pertains to such point. These integrals have already been expressed, equation (53), in a series of polar harmonics, which, if more convergent, would suffice for computing I_1 or I_2. It is easy, however, to derive more convergent and convenient series than that of (53), and this will be the object of the present section.

First, take I_1 of (24). For brevity put

$$w = \frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\beta} \quad \text{and} \quad b = \sin\tfrac{1}{2}\beta.$$

Then by Maclaurin's series, or by the binominal theorem, we readily find

$$I_1 = 2b\int_0^{\frac{1}{2}\pi}(1 - A\sin^2\gamma_1 - B\sin^4\gamma_1 - C\sin^6\gamma_1 - \ldots)\,d\gamma_1, \tag{69}$$

in which

$$\begin{aligned} A &= \tfrac{1}{2}w^2(1-b^2), \\ B &= \tfrac{1}{8}w^4(1+2b^2-3b^4), \\ C &= \tfrac{1}{16}w^6(1+b^2+3b^4-5b^6), \\ &\text{etc.} \end{aligned}$$

The even powers of $\sin \gamma_1$ may each be expanded in a series of the form,

$$c + d \cos 2\gamma_1 + e \cos 4\gamma_1 + \ldots,$$

in which c, d, e, etc. are constants.

But since
$$\int_0^{\frac{1}{2}\pi} \cos 2n\gamma \, d\gamma = 0,$$

we shall need in these expansions only the values of c. The value of c in the expansion of $\sin^{2n}\gamma$ is

$$c = \frac{2n(2n-1)(2n-2)\ldots(n+1)}{1 \cdot 2 \cdot 3 \ldots n} \cdot \left(\frac{1}{2}\right)^{2n}.$$

Applying this formula, and making the integration in (69), there results

$$I = b\pi\left(1 - \tfrac{1}{2}A - \tfrac{3}{8}B - \tfrac{5}{16}C - \ldots\right).$$

Hence if we put*

$$\begin{aligned}
g_1 &= \tfrac{1}{4}(1-b^2),\\
g_2 &= \tfrac{3}{64}(1-b^2)(1+3b^2),\\
g_3 &= \tfrac{5}{256}(1-b^2)(1+2b^2+5b^4),\\
g_4 &= \tfrac{35}{16384}(1-b^2)(5+9b^2+15b^4+35b^6),\\
g_5 &= \tfrac{63}{65536}(1-b^2)(7+12b^2+18b^4+28b^6+63b^8),\\
g_6 &= \tfrac{231}{1048576}(1-b^2)(21+35b^2+50b^4+70b^6+105b^8+231b^{10}),\\
&\text{etc.},
\end{aligned}$$

$$I_1 = b\pi\left(1 - g_1 w^2 - g_2 w^4 - g_3 w^6 - \ldots\right). \tag{70}$$

$$w = \frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\beta}, \quad b = \sin\tfrac{1}{2}\beta, \quad \alpha \leqq \beta.$$

This series converges rapidly except for values of w near unity. In an important practical application, wherein $\beta = 38°$, (70) gives, using

* The general value of g is

$$g_n = \frac{2n(2n-1)(2n-2)\ldots(n+1)}{1^2 \cdot 2^2 \cdot 3^2 \ldots n^2 \cdot 2^{3n}} \left\{ \begin{array}{l}
1 \cdot 1 \cdot 3 \cdot 5 \ldots (2n-3)\\
+ 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2n-5)\, 1 \cdot n b^2\\
+ 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2n-7)\, 1 \cdot 3 \dfrac{n(n-1)}{1 \cdot 2} b^4\\
+ 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2n-9)\, 1 \cdot 3 \cdot 5 \dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} b^6\\
+ \ldots\\
- 1 \cdot 3 \cdot 5 \ldots (2n-1)\, b^{2n}
\end{array} \right\}.$$

terms up to that in w^{12} inclusive, I_1 too great by about 5 per cent. for the case $w = 1$. But this is the most unfavorable case, and one, moreover, for which the exact value of I_1 is known from equation (28).

By a process entirely similar to that followed above, the expansion of (25) gives,* writing for brevity,

$$\nu = \operatorname{cosec} \tfrac{1}{2}a,$$

$$I_2 = \tfrac{1}{2}b^2\pi r \left\{ \begin{array}{l} 1 + \frac{1}{8}b^2 \; (1 + \nu^2) \\ + \frac{1}{64}b^4 \; (3 + 2\nu^2 + 3\nu^4) \\ + \frac{5}{1024}b^6 \; (5 + 3\nu^2 + 3\nu^4 + 5\nu^6) \\ + \frac{7}{16384}b^8 \; (35 + 20\nu^2 + 18\nu^4 + 20\nu^6 + 35\nu^8) \\ + \frac{21}{131072}b^{10} \; (63 + 35\nu^2 + 30\nu^4 + 30\nu^6 + 35\nu^8 + 63\nu^{10}) \\ + \ldots \end{array} \right\}.$$

If in this expression we put

$$\begin{aligned} k_1 &= \tfrac{1}{2} + \tfrac{1}{16}b^2 + \tfrac{3}{128}b^4 + \tfrac{25}{2048}b^6 + \tfrac{245}{32768}b^8 + \tfrac{1323}{262144}b^{10} + \ldots, \\ k_2 &= \tfrac{1}{16} + \tfrac{1}{64}b^2 + \tfrac{15}{2048}b^4 + \tfrac{35}{4096}b^6 + \tfrac{735}{262144}b^8 + \ldots, \\ k_3 &= \tfrac{3}{128} + \tfrac{15}{2048}b^2 + \tfrac{63}{16384}b^4 + \tfrac{315}{131072}b^6 + \ldots, \\ k_4 &= \tfrac{25}{2048} + \tfrac{35}{4096}b^2 + \tfrac{315}{131072}b^4 + \ldots, \\ k_5 &= \tfrac{245}{32768} + \tfrac{735}{262144}b^2 + \ldots, \\ k_6 &= \tfrac{1323}{262144} + \ldots, \end{aligned}$$

etc.,

we find
$$I_2 = b\pi\,(k_1 w^{-1} + k_2 w^{-3} + k_3 w^{-5} + \ldots). \tag{71}$$

$$w = \frac{\sin \frac{1}{2}a}{\sin \frac{1}{2}\beta}, \quad b = \sin \tfrac{1}{2}\beta, \quad a \gtreqless \beta.$$

This series converges somewhat more rapidly than (70). For the case in which $\beta = 38°$, and for the extreme value $w = 1$, using terms to that in w^{-11} inclusive, (71) gives I_2 too small by about 3 per cent.

* The general term of I_2 is

$$\frac{1}{4} \cdot \frac{2n\,(2n-1)\,(2n-2) \ldots (n+2)}{1^2 \cdot 2^2 \cdot 3^2 \ldots n^2 \cdot 2^{3n}} \left\{ \begin{array}{l} 1 \cdot 3 \cdot 5 \ldots (2n-1) \\ + 1 \cdot 3 \cdot 5 \ldots (2n-3)\, 1 \cdot n\nu^2 \\ + 1 \cdot 3 \cdot 5 \ldots (2n-5)\, 1 \cdot 3 \, \frac{n\,(n-1)}{1 \cdot 2}\,\nu^4 \\ + 1 \cdot 3 \cdot 5 \ldots (2n-7)\, 1 \cdot 3 \cdot 5 \, \frac{n\,(n-1)\,(n-2)}{1 \cdot 2 \cdot 3}\,\nu^6 \\ + \ldots \\ + 1 \cdot 3 \cdot 5 \ldots (2n-1)\,\nu^{2n} \end{array} \right\} b^{2n}.$$

For points near the border of the disturbing mass, I_2 may be expressed by a more rapidly converging series than (71). Thus from equation (23)

$$I_2 = \int_0^\beta \sqrt{\left(1 - \frac{\cos\beta - \cos\alpha}{\cos p - \cos\alpha}\right)} \, . \, dp.$$

Let $$\cos\beta - \cos\alpha = 2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta) = 2a.$$

Then

$$I_2 = \int_0^\beta \left(1 - \frac{a}{\cos p - \cos\alpha} - \frac{\frac{1}{2}a^2}{(\cos p - \cos\alpha)^2} - \frac{\frac{1}{3}a^3}{(\cos p - \cos\alpha)^3} - \ldots\right) dp.$$

Now if $$X = \int_0^\beta \frac{dp}{\cos p - \cos\alpha}$$

$$= \frac{1}{\sin\alpha}\log_e \frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)};$$

$$\frac{dX}{d\alpha} = -\sin\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2},$$

$$\frac{d^2X}{d\alpha^2} = -\cos\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2} + 2\sin^2\alpha \int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^3},$$

etc.;

whence $$\int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^2} = -\frac{1}{\sin\alpha}\frac{dX}{d\alpha},$$

$$\int_0^\beta \frac{dp}{(\cos p - \cos\alpha)^3} = \frac{1}{2\sin^2\alpha}\left(\frac{d^2X}{d\alpha^2} - \cot\alpha\frac{dX}{d\alpha}\right),$$

etc.

The integrals in the third and higher terms of the above series are thus seen to depend on the integral in the second term. Making the requisite differentiations we find to terms of the third order inclusive,

$$I_2 = \beta - \left(\frac{a}{\sin\alpha} + \frac{a^2\cos\alpha}{2\sin^3\alpha} + \frac{a^3(3 - 2\sin^2\alpha)}{4\sin^5\alpha} + \ldots\right)\log_e \frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}$$
$$- \left(\frac{5a\sin\beta}{16\sin^2\alpha} + \frac{3a^4\cos\alpha\sin\beta}{8\sin^4\alpha} + \ldots\right), \tag{72}$$

$$a = \sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$

24. Having derived the requisite formulæ for computing the position of any point of the disturbed surface, it remains to determine the slope of this surface relative to the undisturbed surface.

Differentiating equation (64) with respect to a, and dividing the result by the radius of the undisturbed surface r_0, we get

$$\frac{dv}{r_0 da} = \frac{3h\rho}{r_0 \pi \rho_m} \cdot \frac{dI}{da}. \tag{73}$$

This expresses the slope or inclination of the disturbed to the undisturbed surface in a meridian plane through the centre of the disturbing mass; it also expresses the deflection of the plumb-line in the same plane.

In order to apply (73), it is essential to have the general value of

$$\frac{dI}{da}.$$

Since

$$w^n = \left\{\frac{\sin\frac{1}{2}a}{\sin\frac{1}{2}\beta}\right\}^n,$$

$$\frac{dw^n}{da} = \tfrac{1}{2}n\, w^n \cot\tfrac{1}{2}a;$$

and hence (70) gives

$$\frac{dI}{da} = -\pi\cos\tfrac{1}{2}a\left\{\begin{array}{l} 1g_1 w^1 \\ + 2g_2 w^3 \\ + 3g_3 w^5 \\ + 4g_4 w^7 \\ + \ldots \end{array}\right\}. \tag{74}$$

Similarly (71) gives

$$\frac{dI_2}{da} = -\pi\cos\tfrac{1}{2}a\left\{\begin{array}{l} \frac{1}{2}k_1 w^{-2} \\ + \frac{3}{2}k_2 w^{-4} \\ + \frac{5}{2}k_3 w^{-6} \\ + \frac{7}{2}k_4 w^{-8} \\ + \ldots \end{array}\right\}. \tag{75}$$

Equations (74) and (75) will suffice for the computation of dI/da except for points near to or at the border of the attracting mass. As a approaches equality to β the above series become less and less convergent, and finally divergent

when $\alpha = \beta$, or $w = 1$. This may be most readily seen by differentiating (22) or (23) with respect to α and then making $\alpha = \beta$. Thus we find

$$\frac{dI}{d\alpha} = -\tfrac{1}{2}\sin\beta\int_0^\beta \frac{dp}{\cos\beta - \cos p} = -\frac{1}{2}\left[\log\frac{\sin\frac{1}{2}(\beta + p)}{\sin\frac{1}{2}(\beta - p)}\right]_0^\beta = -\infty.$$

Likewise the integrals (24) and (25) become after differentiating them with respect to α and then making $\alpha = \beta$,

$$\frac{dI_1}{d\alpha} = -\int_0^{\frac{1}{2}\pi}\frac{\sec^2\gamma_1\tan\gamma_1\,d\gamma_1}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_1)^3}} \qquad (A)$$

$$\frac{dI_2}{d\alpha} = -\int_0^{\frac{1}{2}\pi}\frac{\sec^2\gamma_2\,d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)}} \qquad (B)$$

$$= -\int_0^{\frac{1}{2}\pi}\frac{\sec^2\frac{1}{2}\beta\sec^2\gamma_2\,d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)^3}} - \int_0^{\frac{1}{2}\pi}\frac{\sec^2\gamma_2\tan^2\gamma_2\,d\gamma_2}{\sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)^3}}$$

$$= -1 + \frac{dI_1}{d\alpha}.$$

This shows the equality of (A) and (B), since (B) is plainly infinite, its value being

$$-\left[\log_e\frac{\tan\gamma_2 + \sqrt{(\sec^2\frac{1}{2}\beta + \tan^2\gamma_2)}}{\sec^2\frac{1}{2}\beta}\right]_0^{\frac{1}{2}\pi}.$$

25. This failure of equations (74) and (75) for points at the border of the attracting mass arises from the fact that the expressions (20) and (21), though very approximate for the magnitude of the potential V, are not sufficiently general to give an accurate value of $dV/d\alpha$, or the attraction in the direction of the arc α, for those points. To determine the slope of the disturbed surface at the immediate border of the disturbing mass a special investigation is requisite.

Since by equations (3) and (6) the slope is expressed by

$$\frac{dv}{r_0 d\alpha} = \frac{3}{4}\cdot\frac{1}{r_0\pi\rho_m}\cdot\frac{dV}{r_0 d\alpha}, \qquad (76)$$

we may derive an expression for the attraction $dV/r_0 d\alpha$ directly. The exact ex-

pression for the horizontal attraction towards the axis of the mass, of any element-mass is, using the same notation as in § 7,

$$-\rho\frac{4r^3\,dr\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\,d\theta\cos\lambda\,d\lambda}{\sqrt{[(r-r')^2+4rr'\sin^2\tfrac{1}{2}\theta]^3}};$$

and the integral of this is dV/r_0da.

Now, as heretofore, let $\qquad r = r_0 + u,$

$$r' = r_0 + v.$$

In addition put $\qquad \eta = r - r',$

so that $\qquad d\eta = dr;$

$$\eta = -v,\quad \text{for}\quad r = r_0,$$
$$= h - v,\quad \text{for}\quad r = r_0 + h.$$

Also let $\qquad \xi = 2r_0\sin\tfrac{1}{2}\theta,$

whence $\qquad d\xi = r_0\cos\tfrac{1}{2}\theta\,d\theta,$

$$\cos\tfrac{1}{2}\theta = \sqrt{\left[1-\left(\frac{\xi}{2r_0}\right)^2\right]}.$$

Making the substitutions and neglecting terms of the order

$$\frac{u}{r_0},\ \frac{v}{r_0},\ \text{and}\ \left(\frac{\xi}{2r_0}\right)^2,$$

the above expression becomes

$$\rho\frac{\xi^2\,d\xi\,d\eta\cos\lambda\,d\lambda}{\sqrt{(\xi^2+\eta^2)^3}}.$$

Integrating with respect to η and substituting the limits given above, there results

$$\left(\frac{(h-v)\,d\xi}{\sqrt{[\xi^2+(h-v)^2]}}+\frac{v\,d\xi}{\sqrt{(\xi^2+v^2)}}\right)\cos\lambda\,d\lambda.$$

If now we suppose the attracted point on the border of the attracting mass, the limits of ξ will be 0, and, with sufficient approximation, $2r_0\sin\beta\cos\lambda = c\cos\lambda$, say. Integrating with respect to ξ and substituting these limits, we get

$$\rho\,(h-v)\cos\lambda\,d\lambda\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{(h-v)^2}\right)}+\frac{c\cos\lambda}{h-v}\right]$$
$$+\rho v\cos\lambda\,d\lambda\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{v^2}\right)}+\frac{c\cos\lambda}{v}\right].$$

It remains to integrate these last expressions with respect to λ between the limits

0 and $\frac{1}{2}\pi$. An application of the formula for integration by parts will readily transform them to elliptics, but since their element functions decrease very rapidly from the lower to the upper limit, the following process* will suffice. Consider the integral

$$\int_0^\lambda \cos\lambda\, d\lambda \log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{v^2}\right)}+\frac{c\cos\lambda}{v}\right],$$

in which λ is such that $\left(\frac{v}{c\cos\lambda}\right)^2$ may be neglected in comparison with unity. In the cases we have to consider $\left(\frac{v}{c\cos\lambda}\right)^2$ will not exceed $\frac{1}{100}$ if $\cos\lambda=\frac{1}{100}$, or $\lambda=89^\circ\ 25'$, about. Then since

$$\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{v^2}\right)}+\frac{c\cos\lambda}{v}\right]$$
$$=\log_e\left[\frac{2c\cos\lambda}{v}\left(1+\frac{1}{4}\left(\frac{v}{c\cos\lambda}\right)^2+\ldots\right)\right],$$

the above integral becomes

$$\int_0^\lambda \cos\lambda\, d\lambda \log_e\frac{2c\cos\lambda}{v}=\log_e\frac{2c}{v}\int_0^\lambda\cos\lambda\, d\lambda+\int_0^\lambda\cos\lambda\, d\lambda\log_e\cos\lambda$$
$$=\sin\lambda\left(\log_e\frac{2c}{v}-1\right)+\log_e(1+\sin\lambda)$$
$$+(\sin\lambda-1)\log\cos\lambda.$$

But since $\sin\lambda$ is very nearly unity, the last expression reduces to

$$\log_e\left(\frac{4c}{v}\right)-1.$$

The error of this integral arising from the use of λ instead of $\frac{1}{2}\pi$ as the upper limit, is less than

$$(\tfrac{1}{2}\pi-\lambda)\cos\lambda\log_e\left[\sqrt{\left(1+\frac{c^2\cos^2\lambda}{v^2}\right)}+\frac{c\cos\lambda}{v}\right],$$

which, if $\cos\lambda=\frac{1}{100}$ and $c\cos\lambda/v=10$, amounts to about $\frac{1}{3000}$.

*Given in a somewhat different form by Helmert in *Theorieen der Höheren Geodäsie*, Vol. II. p. 322.

For the entire attraction, therefore, of the mass for a point on its border we get

$$\frac{dV}{r_0 da} = -2\rho\left[(h-v)\left(\log_e \frac{4c}{h-v} - 1\right) + v\left(\log_e \frac{4c}{v} - 1\right)\right]$$

$$= -2\rho\left[h\left(\log_e \frac{4c}{h-v} - 1\right) + v\log_e \frac{h-v}{v}\right].$$

Finally, restoring in the last expression the value of c, viz. $c = 2r_0 \sin\beta$, (76) becomes

$$\frac{dv}{r_0 da} = -\frac{3}{2}\frac{h\rho}{r_0\pi\rho_m}\left(\log_e \frac{8r_0\sin\beta}{h-v} + \frac{v}{h}\log_e \frac{h-v}{v} - 1\right). \qquad (77)$$

26. Thus far the disturbed surface has been referred to a spherical surface concentric with the earth's centre of gravity before the disturbance arose. In determining the effects of the ice-mass in glacial times, this is the proper surface of reference, since we wish to know the distortion of the sea-level in those times relative to the sea-level in preceding and following epochs. If, however, it is desired to consider the joint effect in disturbing the sea-level of existing masses, like the continents, on the hypothesis that such masses rest on the surface of a centrobaric sphere, a better surface of reference will obviously be the disturbed or existing centre of gravity of the earth. The use of the latter centre will require a slight modification of the preceding formulæ defining the disturbed sea-surface.

To determine the radial displacement of the earth's centre of gravity due to the addition of such a superficial mass as we have considered, it is only necessary to equate the statical moment of that mass to the statical moment of the earth's mass, the moment plane being perpendicular to the axis of the disturbing mass at the undisturbed centre of gravity of the earth. The moment of an elementary ring of angular radius β', measured from the axis of the disturbing mass, is to our order of approximation,

$$2r_0^3 h\rho\pi \sin\beta' \cos\beta' \, d\beta'.$$

Hence if σ denote the displacement sought and M the earth's mass,

$$M\sigma = r_0^3 h\rho\pi \int_0^\beta 2\sin\beta' \cos\beta' \, d\beta'$$

$$= r_0^3 h\rho\pi \sin^2\beta.$$

Therefore, by substitution of the value of M given in equation (3), we find

$$\sigma = \frac{3}{4}h\frac{\rho}{\rho_m}\sin^2\beta. \qquad (78)$$

Now the elevation of any point of the disturbed surface relative to the sphere in the new position will be less than its elevation relative to the sphere in the former position by an amount whose value to the proper degree of approximation is

$$\sigma \cos a,$$

α being as heretofore the angular distance of the point from the axis of the disturbing mass. That is, if v' denote what v becomes by the change in position of the sphere of reference,

$$v' = v - \sigma \cos \alpha.$$

Hence, by virtue of (64) and (78), we find for the equation of the disturbed surface when the sphere of reference is concentric with the disturbed centre of gravity of the earth,

$$v' = 3h \frac{\rho}{\rho_m} \left[\frac{I}{\pi} - \sin^2 \tfrac{1}{2}\beta \left(1 + \cos \alpha \cos^2 \tfrac{1}{2}\beta \right) \right]; \tag{79}$$

and the slope of the disturbed surface with respect to the surface of reference is

$$\frac{dv'}{r_0 d\alpha} = 3h \frac{\rho}{r_0 \pi \rho_m} \left(\frac{dI}{d\alpha} + \tfrac{1}{4}\pi \sin \alpha \sin^2 \beta \right). \tag{80}$$

27. Thus far the thickness of the disturbing mass has been considered uniform. To determine the effect of a mass symmetrical about an axis, but of variable thickness, we may proceed thus: firstly, suppose the effect of the re-arranged water neglected. Then the differential of equation (64) with respect to β gives

$$\frac{dv}{d\beta} d\beta = \frac{3h\rho}{\pi \rho_m} \cdot \frac{d\left(I - \pi \sin^2 \frac{1}{2}\beta\right)}{d\beta} d\beta. \tag{81}$$

This expresses the elevation of the disturbed surface due to an annulus of angular radius β, of angular width $d\beta$, and of height h, the density ρ being uniform. If in this equation we make h a function of β, or write $h = \varphi(\beta)$, and integrate between the proper limits, the result will be the elevation of the disturbed surface due to a mass whose thickness conforms to the law expressed by $\varphi(\beta)$. Calling, for the sake of distinction, the new value of the elevation of the disturbed surface v'', and the proper limits of β, β_1 and β_2, the result of this integration is

$$v'' = \frac{3\rho}{\pi \rho_m} \int_{\beta_1}^{\beta_2} \frac{d\left(I - \pi \sin^2 \frac{1}{2}\beta\right)}{d\beta} \varphi(\beta)\, d\beta. \tag{82}$$

Secondly, if the effect of the re-arranged water be taken into account, we must add to v'' of (82) the following increment obtained from the third member of (66):—

$$\Delta v'' = \frac{9\rho}{2\rho_m} \sum_{i=1}^{i=\infty} \left\{ \frac{f_i(\cos \alpha) \displaystyle\int_{\beta_1}^{\beta_2} \frac{dF_i(\beta)}{d\beta} \varphi(\beta)\, d\beta}{(2i+1)\dfrac{\rho_m}{\rho_w} - 3} \right\}. \tag{83}$$

The equations (82) and (83) assign the effect of any homogeneous mass symmetrical with respect to an axis, subject to the restriction that the maximum thickness of the mass may be neglected in comparison with the radius of the earth.

The integral in (82) depends on, and will in general be, no less complex than I, which is defined by equations (22) to (25). In the application of (83) it is to be observed that by (46)

$$\frac{dF_i(\beta)}{d\beta} = f_i(\cos\beta)\,.\,\sin\beta.$$

SOLUTIONS OF EXERCISES.

94

O is the centre of the circumscribed circle of ABC, and D, E, F the middle points of its sides. Show that

$$OD^2 + OE^2 + OF^2 = 2R'(2R' - r'),$$

where R', r' are the radii of the circumscribed and inscribed circles of the triangle of the feet of the altitudes. [*R. D. Bohannan.*]

SOLUTION.

Let H be the orthocentre, and A', B', C' the feet of the perpendiculars. Since $OD = R\cos A$, etc.,

$$OD^2 + OE^2 + OF^2 = R^2(\cos^2 A + \cos^2 B + \cos^2 C)$$
$$= R^2(1 - 2\cos A\cos B\cos C).$$

But $\quad R = 2R', \quad$ and $\quad r' = A'H\cos A$

$$= BH\cos A\cos C$$
$$= 2R\cos A\cos B\cos C;$$

$$\therefore\quad OD^2 + OE^2 + OF^2 = R^2\left(1 - \frac{r'}{R}\right) = 2R'(2R' - r').$$

[*Marcus Baker.*]